Eq. Lineares de 2ª ordem com coef. ctes não-homogênea

$$a y'' + b y' + c y = G(x) \qquad (NH)$$

$$a y'' + b y' + c y = 0 \qquad (H)$$

Teorema: A solução geral de (NH) é dada por $y = y_p + y_h$, onde y_h é a sol. geral de (H) e y_p é uma sol. particular de (NH).

Método da variação de parâmetros: dadas y_1 e y_2, sol. LI de (H), queremos encontrar $u_1(x)$ e $u_2(x)$ de modo que $y_p = u_1 y_1 + u_2 y_2$ seja sol. de (NH).

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$= \underbrace{(u_1' y_1 + u_2' y_2)}_{=0} + (u_1 y_1' + u_2 y_2')$$

Supondo que $\boxed{u_1' y_1 + u_2' y_2 = 0}$:

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

$$\therefore (NH) \Rightarrow a\left(u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''\right) + b\left(u_1 y_1' + u_2 y_2'\right) + c\left(u_1 y_1 + u_2 y_2\right) = G$$

$$\Rightarrow a u_1' y_1' + a u_2' y_2' + u_1 \underbrace{(a y_1'' + b y_1' + c y_1)}_{=0 \text{ sol. } (H)} + u_2 \underbrace{(a y_2'' + b y_2' + c y_2)}_{=0 \text{ sol. } (H)} = G(x)$$

$$\Rightarrow \boxed{a\left(u_1' y_1' + u_2' y_2'\right) = G(x)}$$

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = \dfrac{G(x)}{a} \end{cases}$$

$$A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \Rightarrow \det(A) \neq 0.$$

Wronskiano

Exemplo: $y'' + y = tg(x)$, $0 < x < \frac{\pi}{2}$.

(H) $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i = \underset{\alpha}{0} \pm \underset{\beta}{1} i$

$\therefore y_h = e^{0x}\left[C_1 \cos(1x) + C_2 \text{sen}(1x)\right] = C_1 \cos(x) + C_2 \text{sen}(x)$

$y_1 = \cos(x)$ e $y_2 = \text{sen}(x)$

$y_1' = -\text{sen}(x)$ e $y_2' = \cos(x)$

$$\begin{cases} \cos(x)\, u_1' + \text{sen}(x)\, u_2' = 0 & (\times \text{sen} x) \quad \circledast \\ -\text{sen}(x)\, u_1' + \cos(x)\, u_2' = \dfrac{tg(x)}{1} & (\times \cos x) \end{cases}$$

$$\begin{cases} \cos(x)\text{sen}(x)\, u_1' + \text{sen}^2(x)\, u_2' = 0 & (1) \\ -\cos(x)\text{sen}(x)\, u_1' + \cos^2(x)\, u_2' = \text{sen}(x) & (2) \end{cases}$$

$\therefore (1) + (2) \Rightarrow \underbrace{\left[\text{sen}^2(x) + \cos^2(x)\right]}_{=1} u_2' = \text{sen}(x) \Rightarrow u_2' = \text{sen}(x)$

$\Rightarrow u_2 = \int \text{sen}(x)\,dx = -\cos(x)$

Substituindo em $\circledast$:

$\cos(x)\, u_1' + \text{sen}^2(x) = 0 \Rightarrow u_1' = -\dfrac{\text{sen}^2(x)}{\cos(x)} \Rightarrow u_1 = -\int \dfrac{\text{sen}^2(x)}{\cos(x)}\,dx$

$= -\int \dfrac{1 - \cos^2(x)}{\cos(x)}\,dx = -\int \dfrac{1}{\cos(x)} - \cos(x)\,dx = -\int \sec(x)\,dx + \int \cos(x)\,dx$

$= -\ln|\sec(x) + tg(x)| + \text{sen}(x)$.

Assim, $y_p = \left[-\ln|\sec(x) + tg(x)| + \text{sen}(x)\right]\cos(x) - \cos(x)\,\text{sen}(x)$ é uma sol. particular de (NH).

Portanto, a sol. geral do (NH) é

$$y = y_p + y_h$$

$$= \left[-\ln|\sec(x) + tg(x)| + sen(x) \right] \cos(x) - \cos(x)\, sen(x) + C_1 \cos(x) + C_2\, sen(x)$$

Exemplo: $y'' + y = tg(x)$, $0 < x < \frac{\pi}{2}$.

(H) $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i = \underset{\alpha}{0} \pm \underset{\beta}{1} i$

$\therefore y_1 = e^{0x}[\cos(x) + i\,sen(x)] = \cos(x) + i\,sen(x)$ → (Não é bom!)

e $y_2 = e^{0x}[\cos(-x) + i\,sen(-x)] = \cos(x) - i\,sen(x)$

são sol. de (H).

$y_1' = -sen(x) + i\cos(x)$ e $y_2' = -sen(x) - i\cos(x)$

$$\begin{cases} [\cos(x) + i\,sen(x)]\,u_1' + [\cos(x) - i\,sen(x)]\,u_2' = 0 \\ [-sen(x) + i\cos(x)]\,u_1' + [-sen(x) - i\cos(x)]\,u_2' = \frac{tg(x)}{1} \end{cases}$$

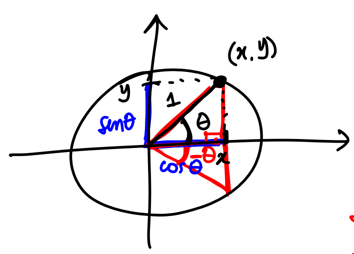
$$\begin{cases} \cos(x)\,(u_1' + u_2') + i\,sen(x)\,(u_1' - u_2') = 0 & (\times \cos(x)) \\ -sen(x)\,(u_1' + u_2') + i\cos(x)\,(u_1' - u_2') = tg(x) & (\times -sen(x)) \end{cases} \quad (I)$$

$$\begin{cases} \cos^2(x)\,(u_1' + u_2') + i\,sen(x)\cos(x)\,(u_1' - u_2') = 0 & (1) \\ sen^2(x)\,(u_1' + u_2') - i\,sen(x)\cos(x)\,(u_1' - u_2') = -tg(x)\,sen(x) & (2) \end{cases}$$

(1) + (2) $\Rightarrow \cos^2(x)(u_1' + u_2') + sen^2(x)(u_1' + u_2') = -tg(x) \cdot sen(x)$

$\Rightarrow (u_1' + u_2')\,\underbrace{(\cos^2(x) + sen^2(x))}_{=1} = -tg(x) \cdot sen(x)$

$\Rightarrow \boxed{u_1' + u_2' = -tg(x) \cdot sen\,x}$



$\cos(\theta) = \frac{x}{1} = x$

$sen(\theta) = \frac{y}{1} = y$

$sen(-\theta) = -sen(\theta)$

$\cos(-\theta) = \cos(\theta)$

$$(\mathrm{I})\begin{cases} \cos(x)(u_1'+u_2') + i\sin(x)(u_1'-u_2') = 0 & (\times \sin(x)) \\ -\sin(x)(u_1'+u_2') + i\cos(x)(u_1'-u_2') = tg(x) & (\times \cos(x)) \end{cases}$$

$$\begin{cases} \sin(x)\cos(x)(u_1'+u_2') + i\sin^2(x)(u_1'-u_2') = 0 & ③ \\ -\sin(x)\cos(x)(u_1'+u_2') + i\cos^2(x)(u_1'-u_2') = \sin(x) & ④ \end{cases}$$

$\therefore$ ③ + ④ $\Rightarrow i\sin^2(x)(u_1'-u_2') + i\cos^2(x)(u_1'-u_2') = \sin(x)$

$\Rightarrow i(u_1'-u_2')\underbrace{(\sin^2 x+\cos^2 x)}_{=1} = \sin x$

$\Rightarrow \boxed{(u_1'-u_2')i = \sin x}$

Assim,

$$\begin{cases} u_1'+u_2' = -tg(x)\cdot\sin(x) & \Rightarrow u_1' = -tg(x)\sin(x) - u_2' \\ (u_1'-u_2')i = \sin(x) \end{cases}$$

$\therefore (-tg(x)\sin(x) - u_2' - u_2')i = \sin(x) \Rightarrow -2u_2' i - tg(x)\sin(x)\, i = \sin(x)$

$\Rightarrow -2u_2' i = \sin x + tg(x)\sin(x)\, i \Rightarrow u_2' = \dfrac{\sin(x) + tg(x)\sin(x)\, i}{-2i}$

$\Rightarrow u_2 = \dfrac{1}{-2i}\displaystyle\int \sin x + tg x \sin x\, i\, dx$

$u_1' = -tg(x)\sin(x) - \underline{\qquad}$

$u_1 = \displaystyle\int \quad \downarrow dx$

$\dfrac{\sin^2}{\cos} = \dfrac{1-\cos^2}{\cos}$

$= \dfrac{1}{\cos} - \cos$

$= \sec$

$\sin^2 + \cos^2 = 1$

$\sin^2 = 1 - \cos^2$